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2004 J. Phys. A: Math. Gen. 37 2331

(http://iopscience.iop.org/0305-4470/37/6/024)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2331–2344 PII: S0305-4470(04)70598-3

The *k*-point random matrix kernels obtained from one-point supermatrix models

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Received 17 October 2003
Published 28 January 2004
Online at stacks.iop.org/JPhysA/37/2331 (DOI: 10.1088/0305-4470/37/6/024)

Abstract

The *k*-point correlation functions of the Gaussian random matrix ensembles are certain determinants of functions which depend on only two arguments. They are referred to as kernels, since they are the building blocks of all correlations. We show that the kernels are obtained, for arbitrary level number, directly from supermatrix models for one-point functions. More precisely, the generating functions of the one-point functions are equivalent to the kernels. This is surprising, because it implies that already the one-point generating function holds essential information about the *k*-point correlations. This also establishes a link to the averaged ratios of spectral determinants, i.e. of characteristic polynomials.

PACS numbers: 05.45.Mt, 03.65.Nk, 05.30.−d

1. Introduction

Random matrix theory (RMT) allows one to model a rich variety of complex systems [1–3]. The Gaussian unitary ensemble (GUE) of random matrices is used in the absence of time reversal invariance. The Gaussian orthogonal ensemble (GOE) and the Gaussian symplectic ensemble (GSE) apply if time-reversal invariance holds and if the levels are not or are Kramers degenerate, respectively. These three cases GOE, GUE and GSE are labelled by the Dyson index $\beta = 1, 2, 4$. Supersymmetry [4, 5] often yields a considerably clearer insight into the structure of random matrix models. This is because supersymmetry drastically reduces the number of degrees of freedom without giving away any information contained in the model. Thus, supersymmetry removes a certain kind of redundancy. Loosely speaking, the supersymmetric formulation plays the role of an 'irreducible representation' for the random matrix model.

In this contribution, we present an unexpected direct connection between the *k*-point correlation functions of the three Gaussian ensembles and the generating functions of the one-point functions. The *k*-point correlation functions are determinants (GUE) or quaternion determinants (GOE, GSE) [1]. All entries of these determinants are fully specified by one function of two energy arguments. Because of their fundamental importance, these functions are referred to as kernels. In the supersymmetric formulation, generating functions are used which, upon derivation with respect to source variables, yield the k-point correlation functions. Particularly the generating function of the one-point correlation function, or rather one-point function, depends on an energy and on a somewhat unphysical source variable. The source variable is needed to break a symmetry in the supersymmetric matrix model. Hence, the number of dependent variables is the same for the generating function of the one-point function and for the kernels. We show in the following that, surprisingly, the generating function of the one-point function is fully equivalent to the kernels. This is true for all three Gaussian ensembles and for arbitrary level numbers. Thus, a fundamental link is established between the one-point functions and the k-point correlation functions.

This paper is organized as follows. For the convenience of the reader, we briefly compile the relevant formulae for the random matrix correlation functions and kernels in section 2. We closely follow Mehta's book [1]. In section 3, we present our main results and discuss implications. The derivations are performed in section 4. Summary and conclusion are given in section 5.

2. Random matrix correlation functions and kernels

The k-point correlation functions $R_k(x_1, \ldots, x_k)$ are the probability densities to find k energies at positions $x_1, \ldots, x_k$, regardless of labelling. They can be written as averages over a probability density $P_N(H)$ of a $N \times N$ Hamilton matrix H,

$$R_k(x_1, \ldots, x_k) = \int P_N(H) \prod_{p=1}^{k} \operatorname{tr} \delta(x_p - H) \, \mathrm{d}[H] \tag{1}$$

where $\mathrm{d}[H]$ is the volume element, that is, the product of the differentials of all independent variables in H. We mention in passing that this definition contains contributions proportional to $\delta(x_p - x_q)$. However, as this issue is not important here, we ignore it and refer the reader to the discussion of those details in [3].

For the Gaussian ensembles, the correlation functions have a remarkable determinant structure [1]. All knowledge needed to construct the full function $R_k^{(\beta)}(x_1, \ldots, x_k)$ is contained in one single function, the kernel, which depends on two energy arguments. In the case of the GUE ($\beta = 2$) for N levels one has

$$R_k^{(2)}(x_1, \ldots, x_k) = \det\left[K_N^{(2)}(x_p, x_q)\right]_{p,q=1,\ldots,k} \tag{2}$$

where the kernel is given by

$$K_N^{(2)}(x_p, x_q) = \sum_{n=0}^{N-1} \varphi_n(x_p)\varphi_n(x_q). \tag{3}$$

Here, $\varphi_n(z)$ denotes the oscillator wavefunction

$$\varphi_n(z) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp\left(-\frac{z^2}{2}\right) H_n(z) \tag{4}$$

and $H_n(z)$ is the Hermite polynomial of order n [6].

Due to the additional symmetries, the corresponding expressions for the GOE ($\beta = 1$) and the GSE ($\beta = 4$) are more involved. For the GOE, the kernel is given by

$$K_N^{(1)}(x_p, x_q) = K_N^{(2)}(x_p, x_q) + \sqrt{\frac{N}{2}} \varphi_{N-1}(x_p) \int_{-\infty}^{+\infty} \varepsilon(x_q - z)\varphi_N(z) \, \mathrm{d}z + \alpha_N(x_p) \tag{5}$$

where $K_N^{(2)}(x_p, x_q)$ is the GUE kernel and the function

$$\alpha_N(x) = \begin{cases} \varphi_{N-1}(x) \Big/ \int_{-\infty}^{+\infty} \varphi_{N-1}(t)\,\mathrm{d}t & N \text{ odd} \\ 0 & N \text{ even} \end{cases} \tag{6}$$

enters. We also use the notation

$$\varepsilon(z) = \tfrac{1}{2}\mathrm{sign}(z). \tag{7}$$

In the case of the GSE, the kernel reads

$$K_N^{(4)}(x_p, x_q) = \frac{1}{\sqrt{2}} K_{2N+1}^{(2)}\left(\sqrt{2}x_p, \sqrt{2}x_q\right) + \sqrt{\frac{2N+1}{2}}\varphi_{2N}\left(\sqrt{2}x_p\right) \int_{-\infty}^{+\infty} \varepsilon(x_q - z)\varphi_{2N+1}(\sqrt{2}z)\,\mathrm{d}z \tag{8}$$

the first two terms are the same as in the GOE kernel, but for $2N + 1$ levels. However, the function $\alpha_N(x)$ does not appear. It is convenient to scale the energy arguments with $\sqrt{2}$.

The ordinary determinant for the GUE correlation functions is replaced by quaternion determinants. One has for the GOE

$$R_k^{(1)}(x_1, \ldots, x_k) = \mathrm{qdet}\begin{bmatrix} K_N^{(1)}(x_p, x_q) & DK_N^{(1)}(x_p, x_q) \\ JK_N^{(1)}(x_p, x_q) & K_N^{(1)}(x_q, x_p) \end{bmatrix}_{p,q=1,\ldots,k} \tag{9}$$

and, similarly, for the GSE

$$R_k^{(4)}(x_1, \ldots, x_k) = \mathrm{qdet}\begin{bmatrix} K_N^{(4)}(x_p, x_q) & DK_N^{(4)}(x_p, x_q) \\ IK_N^{(4)}(x_p, x_q) & K_N^{(4)}(x_q, x_p) \end{bmatrix}_{p,q=1,\ldots,k}. \tag{10}$$

Here D, I and J are certain derivative and integral operators, respectively. We write $K_N^{(2)}(x_p, x_q)$ for the GUE kernel which Mehta denotes by $K_N(x_p, x_q)$ [1]. Mehta works with the kernel $S_N(x_p, x_q)$ for the GOE and the GSE. This function $S_N(x_p, x_q)$ is our $K_N^{(1)}(x_p, x_q)$ without the function $\alpha_N(x_p)$. We decided to introduce the kernels $K_N^{(1)}(x_p, x_q)$ and $K_N^{(4)}(x_p, x_q)$, because the function $\alpha_N(x_p)$ is present only in the GOE, but not in the GSE case. As we will show, the kernels $K_N^{(\beta)}(x_p, x_q)$ are those that appear naturally in the supersymmetry context. More information on the relation between Mehta's kernels and the kernels $K_N^{(\beta)}(x_p, x_q)$ and on how they enter the expressions (9) and (10) for the correlation functions can be found in appendix A.

In concluding this compilation, we underline once more that knowledge of the three kernels suffices to build up all *k*-point correlation functions for the three ensembles GOE, GUE and GSE.

3. Kernels, matrix integrals, generating functions and random matrix averages

Surprisingly, one can obtain the kernels from the lowest-dimensional one-point supermatrix models which reflect the appropriate symmetries. This and its implications state the main result of the present contribution. For the GUE, i.e. for $\beta = 2$, we have

$$K_N^{(2)}(x_q, x_p) = \frac{1}{\pi} \frac{\exp\left(x_p^2/2 - x_q^2/2\right)}{x_p - x_q} \mathrm{Im}\left(\frac{1}{2}\int \exp(-\mathrm{trg}\,\sigma^2)\,\mathrm{detg}^{-N}(\sigma - x^-)\,\mathrm{d}[\sigma] - 1\right) \tag{11}$$

where

$$\sigma = \begin{bmatrix} a & \lambda^* \\ \lambda & \mathrm{i}b \end{bmatrix} \tag{12}$$

is a 2×2 Hermitian supermatrix. The entries a, b are real commuting and λ is complex anticommuting. The energies are ordered in the diagonal matrix $x = \mathrm{diag}(x_p, x_q)$. The variable x_p is supplemented with a small imaginary increment $\mathrm{i}\eta$ such that $x_p^- = x_p - \mathrm{i}\eta$ and $x^- = \mathrm{diag}(x_p^-, x_q)$. The corresponding result in the case of the GOE, i.e. for $\beta = 1$, reads, for even and odd level number N,

$$K_N^{(1)}(x_q, x_p) = \frac{1}{\pi} \frac{\exp\left(x_p^2/2 - x_q^2/2\right)}{x_p - x_q} \times \mathrm{Im}\left(\frac{1}{8}\int \exp\left(-\frac{1}{2}\,\mathrm{trg}\,\sigma^2\right) \mathrm{detg}^{\,-N/2}(\sigma - x^-)\,\mathrm{d}[\sigma] - 1\right). \tag{13}$$

Finally, for the GSE, i.e. for $\beta = 4$, we have

$$K_N^{(4)}(x_q, x_p) = \frac{1}{2\pi} \frac{\exp\left(x_q^2 - x_p^2\right)}{x_q - x_p}\,\mathrm{Im}\left(\frac{1}{8}\int \exp\left(-\mathrm{trg}\,\sigma^2\right) \mathrm{detg}^{\,-N}(\sigma - x^-)\,\mathrm{d}[\sigma] - 1\right). \tag{14}$$

In the cases of the GOE and the GSE, σ is a 4×4 Hermitian supermatrix with an additional symmetry, often referred to as orthosymplectic [4, 5]. Explicitly, σ reads

$$\sigma = \begin{bmatrix} \sqrt{c}a & \sqrt{c}d & \lambda^* & -\lambda \\ \sqrt{c}d & \sqrt{c}b & \mu^* & -\mu \\ \lambda & \mu & \sqrt{-c}w & 0 \\ \lambda^* & \mu^* & 0 & \sqrt{-c}w \end{bmatrix} \tag{15}$$

where $c = 1$ or $c = -1$ for GOE and GSE, respectively. Here the variables a, b, d and w are real commuting, while λ, λ^* and μ, μ^* are complex anticommuting. The diagonal matrix of the energy arguments now also has dimension 4×4 and reads $x = \mathrm{diag}(x_p, x_p, x_q, x_q)$. However, for brevity we always write x for the 2×2 and 4×4 energy matrix. All formulae given here are exact for finite values of N. Thus, as always, supersymmetry decouples the number of integrations to be done from the level number N. The number of integrations is fixed, while the level number N may take arbitrary values. The common form of the three results (11)–(14) is evident. The differences for the three Gaussian ensembles lie in the structure of the matrices σ. Our definitions and notation are the standard ones [4, 5] and, in particular, they follow the definitions and notation of [7–9]. Thus, we may even formulate the results (11)–(14) in the compact form

$$K_N^{(\beta)}(x_q, x_p) = \frac{1}{\gamma\pi} \frac{\exp\left(\gamma\left(x_p^2 - x_q^2\right)/2\right)}{x_p - x_q} \times \mathrm{Im}\left(\frac{\beta^2}{8\gamma^4}\int \exp\left(-\frac{\beta}{2\gamma}\,\mathrm{trg}\,\sigma^2\right) \mathrm{detg}^{\,-\beta N/2|\gamma|}(\sigma - x^-)\,\mathrm{d}[\sigma] - 1\right) \tag{16}$$

where we introduced $\gamma = 1$ for $\beta = 1, 2$ and $\gamma = -2$ for $\beta = 4$.

In spite of its non-trivial character, result (11) is easily proved because it is an immediate consequence of an integral representation of the kernel $K_N^{(2)}(x_p, x_q)$ which was found in [9]. We briefly sketch the derivation in section 4. In fact, expressions similar to equation (11) have already been used for a study involving the chiral GUE [10] and for a certain generalization of the GUE [11]. On the other hand, the proofs of the results (13) and (14) are more involved. They will also be given in section 4.

Formulae (11)–(14) are remarkable, because they establish a direct and previously unknown connection between the kernels and the generating functions $Z_1^{(\beta)}(\bar{x})$ of the one-point functions,

$$\hat{R}_1^{(\beta)}(x_1) = \frac{1}{2|\gamma|\pi}\frac{\partial}{\partial J_1} Z_1^{(\beta)}(\bar{x})\bigg|_{J_1=0}. \tag{17}$$

We introduced the diagonal matrix $\bar{x} = \mathrm{diag}(x_1 - J_1, x_1 + J_1)$ which contains the energy argument x_1 and the source variable J_1. To be consistent with the previous notation, we use x_1 to denote the argument of the one-point functions. The one-point function is written as $\hat{R}_1^{(\beta)}(x_1) = \tilde{R}_1^{(\beta)}(x_1) + \mathrm{i}R_1^{(\beta)}(x_1)$ such that the level density is the imaginary part, $\mathrm{Im}\,\hat{R}_1^{(\beta)}(x_1) = R_1^{(\beta)}(x_1)$. As is well known, the matrix integrals in the expressions (11)–(14) are precisely the generating functions,

$$Z_1^{(\beta)}(x) = \frac{\beta^2}{8\gamma^4}\int \exp\left(-\frac{\beta}{2|\gamma|}\,\mathrm{trg}\,\sigma^2\right) \mathrm{detg}^{\,-\beta N/2|\gamma|}(\sigma - x^-)\,\mathrm{d}[\sigma] \tag{18}$$

with the appropriate supermatrices σ. Thus, we arrive at

$$\begin{aligned}
K_N^{(2)}(x_q, x_p) &= \frac{1}{\pi}\exp\left(x_p^2/2 - x_q^2/2\right)\mathrm{Im}\,\frac{Z_1^{(2)}(x) - Z_1^{(2)}(0)}{x_p - x_q}\\
K_N^{(1)}(x_q, x_p) &= \frac{1}{\pi}\exp\left(x_p^2/2 - x_q^2/2\right)\mathrm{Im}\,\frac{Z_1^{(1)}(x) - Z_1^{(1)}(0)}{x_p - x_q}\\
K_N^{(4)}(x_q, x_p) &= \frac{1}{2\pi}\exp\left(x_q^2 - x_p^2\right)\mathrm{Im}\,\frac{Z_1^{(4)}(x) - Z_1^{(4)}(0)}{x_q - x_p}.
\end{aligned} \tag{19}$$

For the GSE kernel $K_N^{(4)}(x_q, x_p)$ the arguments of the exponential are interchanged with respect to the GUE kernel $K_N^{(2)}(x_q, x_p)$ and the GOE kernel $K_N^{(1)}(x_q, x_p)$. We note that $Z_1^{(\beta)}(x)$ depends on the two energies x_p and x_q. There is no source variable here. Moreover, we have $Z_1^{(\beta)}(0) = 1$ due to the definition of the generating function. Again, we can write

$$K_N^{(\beta)}(x_q, x_p) = \frac{1}{\gamma\pi}\exp\left(\frac{\gamma}{2}\left(x_p^2 - x_q^2\right)\right)\mathrm{Im}\,\frac{Z_1^{(\beta)}(x) - Z_1^{(\beta)}(0)}{x_q - x_p} \tag{20}$$

which combines the three results (19) in a compact form.

Formulae (19) state a close connection between the kernels and the generating functions. The kernels can be viewed as difference quotients of the generating functions at the two points x and 0. The crucial quantity is the difference $x_p - x_q$. By construction, the generating functions are unity whenever the two arguments degenerate. Thus, $Z_1^{(\beta)}(x)$ moves away from unity as function of $x_p - x_q$. If one takes the limit $x_q \to x_p$, the difference quotient becomes the differential quotient (17). This yields the kernels at $x_p = x_q$, that is, the level densities as function of the single remaining variable. In appendix B we discuss extensions of the previous results if real parts contribute to the correlation functions.

To further clarify the meaning of these findings, we rewrite the generating functions as averages over the original random matrices in ordinary space,

$$Z_1^{(\beta)}(x) = C_{N\beta}\int \exp\left(-\frac{\beta}{2}\,\mathrm{tr}\,H^2\right)\left(\frac{\det(H - x_q)}{\det(H - x_p^-)}\right)^{|\gamma|}\mathrm{d}[H] \tag{21}$$

with normalization constants $C_{N\beta}$. Here, the matrices H parametrize the GOE, GUE and GSE for $\beta = 1, 2, 4$, respectively, as defined in Mehta's book [1]. Combining equations (19) and (21), we see that the kernels themselves are, apart from factors, averages over the Gaussian ensembles. This is a surprising insight. According to the definition (1), the k-point correlation function of a Gaussian ensemble is one single matrix integral for a fixed value of k. The results presented here imply that this single average breaks up into products of averages. This is intimately related to the determinant structure, but it is a stronger statement because it identifies the determinant structure as stemming from the break-up of the random matrix average.

Furthermore, it follows from equations (19) and (21) that the random matrix kernels are essentially an average over a ratio of spectral determinants, taken at the two different energies. This relates our findings to the presently much discussed issue of characteristic polynomials, spectral determinants and their moments (see [12, 13] and references therein). For matrix dimension $N = 2$, the connection between averages over the ratio of two characteristic polynomials and the kernel was recently observed in [14] in the GOE case ($\beta = 1$).

4. Kernels expressed as eigenvalue integrals in superspaces

We prove the results in the previous section by explicit calculation. Alternatively, one could try to employ Dyson's Brownian motion [1] and its supersymmetric extension [15] for the stationary case, i.e. for the pure ensembles. However, this would still leave one with the problem of fixing the boundary conditions in an unambiguous way. Another strategy could consist in showing that the supermatrix models satisfy the same equations that the kernels obey, such as the convolution condition. Once again, one is confronted with some ambiguity. Thus, we believe that the most direct proof is probably an explicit calculation, but we certainly do not exclude that other direct proofs also exist.

There are two possibilities to proceed with an explicit calculation. First, due to the small dimensions of the supermatrices in equation (22), one can expand the superdeterminants in the supermatrix models and integrate out the Grassmann variables by 'brute force'. The resulting expressions are rather complicated and the calculations to follow are quite cumbersome. Second, one can introduce eigenvalue–angle coordinates and integrate in a first step over the supergroups and in a second one over the eigenvalues. We present this approach in the following because the eigenvalue integrals to be solved here are of a general type which will always appear in exact calculations involving supersymmetry. In particular, they will show up in generalizations of the present supermatrix models. Thus, we want to develop techniques for how to handle them.

We denote the right-hand sides of the formulae (11)–(14) by

$$L_N^{(\beta)}(x_p, x_q) = \frac{\sqrt{|\gamma|}}{\gamma\pi}\frac{1}{x_p - x_q} \times \operatorname{Im}\left(\frac{\beta^2}{8\gamma^4}\int \exp\left(-\frac{\beta}{2|\gamma|}\operatorname{trg}\sigma^2\right)\operatorname{detg}^{-\beta N/2|\gamma|}(\sigma - x^-)\,\mathrm{d}[\sigma] - 1\right) \tag{22}$$

where, for notational convenience, the exponential functions are not included. For the same reason, we also split off a factor $\sqrt{\gamma}$. The strategy for the ensuing calculation is now the same in all three cases $\beta = 1, 2, 4$. The supermatrix integral yields two contributions, an Efetov–Wegner term [16] and an eigenvalue integral. The Efetov–Wegner term gives a constant, cancelling the unity to be subtracted in the bracket. To obtain the eigenvalue integral, we shift the integration matrix σ by the diagonal matrix x, introduce eigenvalue–angle coordinates $\sigma = u^{-1}su$ for the shifted integration matrix and perform the group integral over the diagonalizing matrix u. For $\beta = 2$, the group integral is the well known supersymmetric Itzykson–Zuber integral [9]. It is trivial here, because u is only a 2×2 supermatrix. For $\beta = 1, 4$ the group integral is over 4×4 supermatrices u and thus non-trivial. In both cases

$\beta = 1, 4$, it is the same group integral which can be viewed as a supersymmetric extension of Gelfand's spherical functions. This integral was first calculated in [7, 8]. (Responding to some confusion in the literature, we mention in passing that the integrals involving 8×8 supermatrices were also first solved in these references. However, in the present case, the 4×4 case suffices.) After the group integrations, we are left with integrals over the eigenvalues $s = \mathrm{diag}(s_1, \mathrm{i}s_2)$ for $\beta = 2$, $s = \mathrm{diag}(s_{11}, s_{21}, \mathrm{i}s_2, \mathrm{i}s_2)$ for $\beta = 1$ and $s = \mathrm{diag}(s_1, s_1, \mathrm{i}s_{12}, \mathrm{i}s_{22})$ for $\beta = 4$. These eigenvalue integrals are solved in the sequel. We note that the eigenvalues in the fermion–fermion block of the supermatrix σ carry an imaginary unit. It is due to a Wick rotation which is necessary to ensure convergence of the supermatrix integrals. Thus, this imaginary unit has to be ignored when taking the imaginary part on the right-hand side of equation (22).

4.1. Gaussian unitary ensemble

The energy difference $x_p - x_q$ drops out and the matrix integral reduces to the double integral

$$L_N^{(2)}(x_p, x_q) = -\frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\mathrm{d}s_1\,\mathrm{d}s_2}{s_1 - \mathrm{i}s_2} \exp(-(s_1 + x_p)^2 + (\mathrm{i}s_2 + x_q)^2)(\mathrm{i}s_2)^N \,\mathrm{Im}\, \frac{1}{(s_1^-)^N}. \tag{23}$$

This coincides with the double integral found in [9]. In this reference, it was denoted by $C_N(x_p, x_q)$. However, it is important to note that the double integral resulted in [9] from calculating the k-point correlation function for arbitrary k, that is, from a group integral over a $2k \times 2k$ unitary supermatrix. In [9], the double integral was already evaluated and it was shown that

$$K_N^{(2)}(x_p, x_q) = \exp\left(x_p^2/2 - x_q^2/2\right) L_N^{(2)}(x_p, x_q). \tag{24}$$

This proves formula (11).

4.2. Gaussian orthogonal ensemble

The orthogonal case has a much more complicated structure. From [7, 8], a triple integral results,

$$\begin{aligned} L_N^{(1)}(x_p, x_q) = {} & \frac{1}{8\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{|s_{11} - s_{21}|\,\mathrm{d}s_{11}\,\mathrm{d}s_{21}\,\mathrm{d}s_2}{(s_{11} - \mathrm{i}s_2)^2 (s_{21} - \mathrm{i}s_2)^2} \\ & \times \exp\left(-\frac{1}{2}(s_{11} + x_p)^2 - \frac{1}{2}(s_{21} + x_p)^2 + (\mathrm{i}s_2 + x_q)^2\right) \\ & \times (2(x_p - x_q)(s_{11} - \mathrm{i}s_2)(s_{21} - \mathrm{i}s_2) + (s_{11} + s_{21} - 2\mathrm{i}s_2)) \\ & \times (\mathrm{i}s_2)^N \,\mathrm{Im}\, \frac{1}{(s_{11}^-)^{N/2} (s_{21}^-)^{N/2}}. \end{aligned} \tag{25}$$

Again, the energy difference $x_p - x_q$ in the denominator has been cancelled. The technical difficulty to overcome is twofold. First, the integration variables are coupled in an inconvenient way, even involving an absolute value. This happens in the term in front of the Gaussians which stems from the Jacobian or Berezinian of the eigenvalue–angle coordinates. Second, the last term contains an imaginary part of the product of two singularities. It cannot be interpreted as the product of imaginary parts which would simply yield derivatives of δ functions. Both difficulties can be circumvented by observing that the GOE kernel contains, according to equation (5), the GUE kernel. Thus, we split off the latter. To this end, we reformulate the

GUE kernel (23) as a triple integral. We write $s_1 = s_{11}$ and introduce a dummy integration over the variable s_{21} by multiplying the integrand with the function $\delta(s_{11} - s_{21})$.

$$\begin{aligned} L_N^{(2)}(x_p, x_q) = \frac{1}{2\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\delta(s_{11} - s_{21})\, \mathrm{d}s_{11}\, \mathrm{d}s_{21}\, \mathrm{d}s_2}{(s_{11} - \mathrm{i}s_2)(s_{21} - \mathrm{i}s_2)} \\ \times \exp\left(-\frac{1}{2}(s_{11} + x_p)^2 - \frac{1}{2}(s_{21} + x_p)^2 + (\mathrm{i}s_2 + x_q)^2\right) \\ \times (s_{11} + s_{21} - 2\mathrm{i}s_2)(\mathrm{i}s_2)^N \,\mathrm{Im}\, \frac{1}{(s_{11}^-)^{N/2}(s_{21}^-)^{N/2}} \end{aligned} \tag{26}$$

In order to subtract this formula from equation (25) we have to do some integrations by parts in both the expressions. More precisely, we use

$$\delta(s_{11} - s_{21}) = \frac{1}{2}\left(\frac{\partial}{\partial s_{11}} - \frac{\partial}{\partial s_{21}}\right) \varepsilon(s_{11} - s_{21}) \tag{27}$$

in equation (26). This procedure casts the integrand in equation (26) into the adequate form to be subtracted from the left-hand side of equation (25). It is also convenient to do an integration by parts in equation (25) using

$$\begin{aligned} \left(2(x_q - x_p) - \frac{\partial}{\partial \mathrm{i}s_2} - \frac{\partial}{\partial s_{11}} - \frac{\partial}{\partial s_{21}} + 2\mathrm{i}s_2 - s_{11} - s_{21}\right) \\ \times \exp\left(-\frac{1}{2}(s_{11} + x_p)^2 - \frac{1}{2}(s_{21} + x_p)^2 + (\mathrm{i}s_2 + x_q)^2\right) = 0. \end{aligned} \tag{28}$$

With these adjustments we can subtract $L_N^{(2)}(x_p, x_q)$ from $L_N^{(1)}(x_p, x_q)$ and obtain

$$\begin{aligned} M_N^{(1)}(x_p, x_q) &= L_N^{(1)}(x_p, x_q) - L_N^{(2)}(x_p, x_q) \\ &= \frac{N}{8\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |s_{11} - s_{21}|\, \mathrm{d}s_{11}\, \mathrm{d}s_{21} \exp\left(-\frac{1}{2}(s_{11} + x_p)^2 - \frac{1}{2}(s_{21} + x_p)^2\right) \\ &\quad \times \mathrm{Im}\, \frac{1}{(s_{11}^-)^{N/2+1}(s_{21}^-)^{N/2+1}} \int_{-\infty}^{+\infty} \mathrm{d}s_2\ \exp(\mathrm{i}s_2 + x_q)^2 (\mathrm{i}s_2)^{N-1}. \end{aligned} \tag{29}$$

This expression decouples the three-dimensional integral into a product of a two-dimensional integral and a one-dimensional integral. Furthermore, the left-hand side of equation (29) factorizes into a product of functions each depending only on one energy argument. Therefore, we can write

$$M_N^{(1)}(x_p, x_q) = \frac{N}{8\pi^2} \omega_N^{(1)}(x_p) \psi_N^{(1)}(x_q) \tag{30}$$

The function $\psi_N^{(1)}$ is simply an integral representation for the Hermite polynomial. The integration of $\omega_N^{(1)}$ requires more effort. The relations

$$\begin{aligned} -\left(\frac{N}{2} + 1\right)\left(\frac{\partial}{\partial x_p} + 2x_p\right) \omega_{N+2}^{(1)}(x_p) = \frac{\partial}{\partial x_p} \omega_N^{(1)}(x_p) \\ \omega_N^{(1)}(x_p) - \left(\frac{N}{2} + 1\right) \omega_{N+2}^{(1)}(x_p) = 4\pi \frac{(-1)^{N+1}}{(N+1)!} H_{N+1}(x_p) \exp\left(-x_p^2\right) \end{aligned} \tag{31}$$

are used. The second formula above was derived by using

$$H_N(x_p) = \frac{(-1)^N N!}{\pi} \exp\left(x_p^2\right) \mathrm{Im} \int_{-\infty}^{\infty} \frac{\exp(-(\xi + x)^2)}{(\xi^-)^{N+1}}\, \mathrm{d}\xi \tag{32}$$

and by the introduction of a dummy variable similar to equation (26). By combining these relations, we obtain

$$\omega_N^{(1)}(x_p) = -\exp\left(-x_p^2/2\right)\left(\frac{4\pi(-1)^N}{N!}\int_{-\infty}^{\infty}\varepsilon(x_p-t)H_N(t)\exp\left(-\frac{t^2}{2}\right)\,\mathrm{d}t + c_N^{(1)}\right)$$
$$\psi_N^{(1)}(x_q) = \frac{\sqrt{\pi}(-1)^{N-1}}{2^{N-1}}H_{N-1}(x_q). \tag{33}$$

The form of the above equations are the expected ones, and the remaining problem is to calculate the integration constant $c_N^{(1)}$. This tedious calculation is performed in appendix C. We find

$$c_N^{(1)} = \begin{cases} 0 & \text{if} \quad N \text{ even} \\ -4\pi\, 2^{N/2}/N!! & \text{if} \quad N \text{ odd.} \end{cases} \tag{34}$$

This non-vanishing constant gives rise to a contribution which is identified with the function α_N defined in equation (6).

Now, equation (33) is rewritten in terms of the oscillator wavefunctions φ_n defined in (4) and compared to (3) and (5). We then have

$$M_N^{(1)}(x_p, x_q) = \exp\left(x_q^2/2 - x_p^2/2\right)\left(K_N^{(1)}(x_q, x_p) - K_N^{(2)}(x_q, x_p)\right) \tag{35}$$

for all values of N. This proves equation (13).

4.3. Gaussian symplectic ensemble

For the GSE the structure of the supermatrix σ is almost the same as for the GOE. The group integral found in [7, 8] can be applied again. However, boson–boson block and fermion–fermion block are interchanged with respect to the GOE. As a consequence the imaginary unit now comes in front of the integration variables s_{11}, s_{21} and the contribution of the superdeterminant in equation (25) is inverted. With an additional rescaling $\sigma \to \sigma/\sqrt{2}$ we obtain

$$\begin{aligned} L_N^{(4)}(x_p, x_q) = {} & \frac{1}{8\pi^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{|s_{11}-s_{21}|\,\mathrm{d}s_{11}\,\mathrm{d}s_{21}\,\mathrm{d}s_2}{(\mathrm{i}s_{11}-s_2)^2(\mathrm{i}s_{21}-s_2)^2} \\ & \times\exp\left(\frac{1}{2}\left(\mathrm{i}s_{11}+\sqrt{2}x_p\right)^2 + \frac{1}{2}\left(\mathrm{i}s_{21}+\sqrt{2}x_p\right)^2 - \left(s_2+\sqrt{2}x_q\right)^2\right) \\ & \times\left(2\left(\sqrt{2}x_q - \sqrt{2}x_p\right)(\mathrm{i}s_{11}-s_2)(\mathrm{i}s_{21}-s_2)\right. \\ & \left. + (\mathrm{i}s_{11}+\mathrm{i}s_{21}-2s_2)\right)(\mathrm{i}s_{11}\mathrm{i}s_{21})^N\,\mathrm{Im}\,\frac{1}{(s_2^-)^{2N}}. \end{aligned} \tag{36}$$

Now we can apply the same method as in the case of the GOE. We arrive at the decomposition

$$L_N^{(4)}(x_p, x_q) = L_{2N}^{(2)}\left(\sqrt{2}x_q, \sqrt{2}x_p\right) + M_N^{(4)}(x_p, x_q) \tag{37}$$

$$M_N^{(4)}(x_p, x_q) = \frac{2N}{8\pi^2}\omega_N^{(4)}(x_p)\psi_N^{(4)}(x_q). \tag{38}$$

The functions $\omega_N^{(4)}$ and $\psi_N^{(4)}$ are now given by

$$\begin{aligned} \psi_N^{(4)}(x_q) &= \int_{-\infty}^{+\infty}\mathrm{d}s_2\exp\left(-\left(s_2+\sqrt{2}x_q\right)^2\right)\mathrm{Im}\,\frac{1}{(s_2^-)^{2N+1}} \\ \omega_N^{(4)}(x_p) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathrm{d}s_{11}\,\mathrm{d}s_{21}\,|s_{11}-s_{21}| \\ & \quad \times\exp\left(\frac{1}{2}\left(\mathrm{i}s_{11}+\sqrt{2}x_p\right)^2 + \frac{1}{2}\left(\mathrm{i}s_{21}+\sqrt{2}x_p\right)^2\right)(\mathrm{i}s_{11}\mathrm{i}s_{21})^{N-1} \end{aligned} \tag{39}$$

The function $\psi_N^{(4)}(x_q)$ is easily evaluated by making use of the identity equation (32) for Hermite polynomials

$$\psi_N^{(4)}(x_q) = \frac{\pi}{(2N)!} \exp\left(-2x_q^2\right) H_{2N}\left(\sqrt{2}x_q\right). \tag{40}$$

The integration of $\omega_N^{(4)}(x_p)$ is a little more tricky. The procedure follows ideas analogous to those used for the calculation of $\omega_N^{(1)}(x_p)$ for the GOE in section (4.2) and appendix C. An integration constant occurs in this case as well. It can be fixed in the same manner as for the GOE. Here, however, it vanishes for all values of N. One finds

$$\omega_N^{(4)}(x_p) = -\frac{\sqrt{\pi}}{2^{2N-3}} \exp\left(x_p^2\right) \int_{-\infty}^{+\infty} \varepsilon\left(\sqrt{2}x_p - t\right) \exp(-t^2/2) H_{2N-1}(t)\, \mathrm{d}t. \tag{41}$$

Inserting these results into equation (37) and expressing everything in terms of oscillator wavefunctions one arrives at

$$L_N^{(4)}(x_p, x_q) = \exp\left(x_p^2 - x_q^2\right) \left(\sum_{n=0}^{2N-1} \varphi_n\left(\sqrt{2}x_p\right)\varphi_n\left(\sqrt{2}x_q\right) + \sqrt{\frac{2N}{2}}\varphi_{2N}\left(\sqrt{2}x_q\right) \right.$$
$$\left. \times \int_{-\infty}^{+\infty} \varepsilon\left(\sqrt{2}x_p - t\right)\varphi_{2N-1}(t)\, \mathrm{d}t \right). \tag{42}$$

This is almost the final result. We use the integration formula

$$\sqrt{\frac{2N}{2}} \int_{-\infty}^{+\infty} \varepsilon(x - t)\varphi_{2N-1}(t)\, \mathrm{d}t = \varphi_{2N}(x) + \sqrt{\frac{2N+1}{2}} \int_{-\infty}^{+\infty} \varepsilon(x - t)\varphi_{2N+1}(t)\, \mathrm{d}t \tag{43}$$

to obtain

$$L_N^{(4)}(x_p, x_q) = \exp\left(x_p^2 - x_q^2\right) \left(K_{2N+1}^{(2)}\left(\sqrt{2}x_p, \sqrt{2}x_q\right) \right.$$
$$\left. + \sqrt{\frac{2N+1}{2}}\varphi_{2N}\left(\sqrt{2}x_q\right) \int_{-\infty}^{+\infty} \varepsilon\left(\sqrt{2}x_p - t\right)\varphi_{2N+1}(t)\, \mathrm{d}t \right). \tag{44}$$

This is exactly our assertion (14).

5. Summary and conclusion

We showed that the generating functions for the one-point functions directly yield the kernels of the correlation functions in RMT. This is tantamount to saying that the kernels are given by the lowest-dimensional supermatrix models. We proved this by explicit calculations for the Gaussian ensembles GOE, GUE and GSE. Recent results for supergroup integrals enter our derivation. We develop new techniques for integrals over eigenvalues of supermatrices. This was another reason for us to prove our results by explicit calculation.

The equivalence between kernels and the generating functions for the one-point functions is an unexpected, surprising insight. The generating functions contain a symmetry, breaking the source term. Our results demonstrate that this symmetry breaking is intimately related to the correlations themselves. Hence, the generating functions comprise much more information than just the one-point functions. Technically, this becomes apparent in the fact that the source variable adds, with different signs, to the energy variable. There are effectively two energy arguments which are then identified with those of the kernels. The kernels are obtained as the difference quotient of the generating functions. In the limit of the differential quotient, one finds the well-known relation between one-point functions and their generating functions.

All our results hold for arbitrary level number. Among other things, this opens yet another possibility to calculate the kernels in the limit of large level number on the local scale. Here, one can do that by a saddlepoint approximation of the one-point supermatrix models. No Goldstone modes are present and one finds the kernels for all correlations from the saddlepoints. As this is a straightforward exercise, we have not presented it in this contribution.

Our findings are likely to have further extensions. From [15], one easily concludes that the structural relation we observed carries over, for $\beta = 2$, to models in which a fixed matrix is added to the random matrices. Further investigations are in progress for the cases $\beta = 1$ and $\beta = 4$. Our results could have relevance for field theory as well.

Acknowledgments

TG and HK acknowledge financial support from the Swedish Research Council and from the RNT Network of the European Union with grant no HPRN–CT–2000-00144, respectively. HK also thanks the division of Mathematical Physics, LTH, for its hospitality during his visits to Lund.

Appendix A. Mehta's kernels and the kernels $K_N^{(\beta)}(x_p, x_q)$

The fundamental piece in Mehta's notation is the function

$$S_N(x_p, x_q) = K_N^{(2)}(x_p, x_q) + \sqrt{\frac{N}{2}}\varphi_{N-1}(x_p) \int_{-\infty}^{+\infty} \varepsilon(x_q - z)\varphi_N(z)\, \mathrm{d}z. \qquad \text{(A.1)}$$

Therefore in his expressions for the GOE correlation functions $\alpha_N(x)$ appears as an independent quantity

$$R_k^{(1)}(x_1, \ldots, x_k) = \operatorname{qdet} \begin{bmatrix} S_N(x_p, x_q) + \alpha(x_p) & DS_N(x_p, x_q) \\ JS_N(x_p, x_q) & S_N(x_q, x_p) + \alpha(x_q) \end{bmatrix}_{p,q=1,\ldots,k}. \qquad \text{(A.2)}$$

In the GSE result, the function $\alpha(z)$ does not appear,

$$R_k^{(4)}(x_1, \ldots, x_k) = \operatorname{qdet} \left[\frac{1}{\sqrt{2}} \begin{bmatrix} S_{2N+1}(\sqrt{2}x_p, \sqrt{2}x_q) & DS_{2N+1}(\sqrt{2}x_p, \sqrt{2}x_q) \\ IS_{2N+1}(\sqrt{2}x_p, \sqrt{2}x_q) & S_{2N+1}(\sqrt{2}x_q, \sqrt{2}x_p) \end{bmatrix} \right]_{p,q=1,\ldots,k}. \qquad \text{(A.3)}$$

The operators D, I and J are defined as acting on the function $S(x_p, x_q)$ only,

$$\begin{aligned} DS_N(x_p, x_q) &= -\frac{\mathrm{d}}{\mathrm{d}x_q} S_N(x_p, x_q) \\ IS_N(x_p, x_q) &= \int \mathrm{d}t\, \varepsilon(x_p - t) S_N(t, x_q) \\ JS_N(x_p, x_q) &= IS_N(x_p, x_q) + \int_0^{x_p} \alpha(t)\, \mathrm{d}t - \int_0^{x_q} \alpha(t)\, \mathrm{d}t + \varepsilon(x_p - x_q). \end{aligned} \qquad \text{(A.4)}$$

In our approach, the kernels $K_N^{(\beta)}$, i.e. the complete upper left entries of the 2×2 matrices in equations (A.2) and (A.3), are the fundamental quantities, rather than Mehta's kernels.

Therefore the operators in the off diagonal elements should also be defined as acting on $K_N^{(\beta)}$. This is accomplished by the following definitions:

$$
\begin{aligned}
DK_N^{(\beta)}(x_p, x_q) &= \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x_p} K_N(x_q, x_p) - (x_p \leftrightarrow x_q)\right) \\
IK_N^{(\beta)}(x_p, x_q) &= \frac{1}{2}\left(\int \mathrm{d}t\,\varepsilon(x_p - t) K_N^{(\beta)}(t, x_q) - (x_p \leftrightarrow x_q)\right) \\
JK_N^{(\beta)}(x_p, x_q) &= IK_N(x_p, x_q) + \varepsilon(x_p - x_q).
\end{aligned}
\tag{A.5}
$$

With these definitions our expressions for the correlation functions, equations (9) and (10), are identical with equations (A.2) and (A.3). This is easily verified by using that $DS_N(x_p, x_q)$ and $IS_N(x_p, x_q)$ are antisymmetric in their arguments [1]. We just remark that the simplicity of the definitions (A.5) is another strong hint that the functions $K_N^{(\beta)}(x_p, x_q)$ rather than $S(x_p, x_q)$ are the fundamental quantities.

Appendix B. Real part contributions to the correlation functions

The correlation functions $R_k(x_1, \ldots, x_k)$ in classical RMT are, according to equation (1), averages involving only the imaginary parts of the Green functions. Including the real parts, one has the more general correlation functions

$$
\hat{R}_k(x_1, \ldots, x_k) = \frac{1}{\pi^k} \int P_N(H) \prod_{p=1}^{k} \mathrm{tr}\, \frac{1}{H - x_p}\, \mathrm{d}[H]. \tag{B.1}
$$

We use the notation of [15], cf equation (17). As for the definition (1), we ignore contributions proportional to $\delta(x_p - x_q)$. In the case of the GUE, it has been shown in [9] that the functions (B.1) also have a determinant structure. We conjecture that the quaternion determinant structure carries over to the GOE and the GSE cases, too. The corresponding kernels $\hat{K}_N^{(\beta)}(x_q, x_p)$ are generalizations of the kernels $K_N^{(\beta)}(x_q, x_p)$. We expect that they are given by

$$
\begin{aligned}
\hat{K}_N^{(\beta)}(x_q, x_p) = {} & \frac{1}{\gamma\pi} \frac{\exp\left(\gamma\left(x_p^2 - x_q^2\right)/2\right)}{x_p - x_q} \\
& \times \left(\frac{\beta^2}{2\gamma^4} \int \exp\left(-\frac{\beta}{2|\gamma|}\, \mathrm{trg}\, \sigma^2\right) \mathrm{detg}^{\,-\beta N/2|\gamma|}(\sigma - x^-)\, \mathrm{d}[\sigma] - 1\right)
\end{aligned}
\tag{B.2}
$$

such that equation (16) results when taking the imaginary part. In the GUE case $\beta = 2$, formulae (B.2) is a immediate consequence of [9]. For the GOE and GSE cases $\beta = 1$ and $\beta = 4$, formulae (B.2) states a conjecture. The kernel $\hat{K}_N^{(2)}(x_q, x_p)$ follows from $K_N^{(2)}(x_q, x_p)$ by simply replacing one of the oscillator wave-functions $\varphi_n(z)$ with $\hat{\varphi}_n(z)$. The latter function combines the two independent solutions of the oscillator waveequation, i.e. the function $\varphi_n(z)$ and its Cauchy or Stiltjes transform. To the best of our knowledge, the relevance of those second solutions in an RMT context was first observed in [17]. Again, we conjecture that these features also carry over to the kernels $\hat{K}_N^{(\beta)}(x_q, x_p)$ and $K_N^{(\beta)}(x_q, x_p)$ for $\beta = 1$ and $\beta = 4$.

Appendix C. Calculation of some integration constants

Considering equation (31) at $x_p = 0$, we obtain the recursion formula

$$
c_N^{(1)} - \left(\frac{N}{2} + 1\right) c_{N+2}^{(1)} = \frac{4\pi(-1)^{N+1}}{(N+1)!} \left(H_{N+1}(0) + N b_N - \frac{1}{2} b_{N+2}\right) \tag{C.1}
$$

where

$$b_N = \int_{-\infty}^{\infty} \varepsilon(t)\exp(-t^2/2)H_N(t)\,\mathrm{d}t. \tag{C.2}$$

The right-hand side of equation (C.1) turns out to be zero for all N. This is easily seen for even N. For odd N, one has to employ equation (27) and to integrate by parts. The recursion formula obtained in this way is equivalent to a result given by Mehta [1]. The remaining task is to find $c_0^{(1)}$ and $c_1^{(1)}$ as starting values for an induction.

We employ the implicit definition of $\omega_1^{(1)}(x_p)$ according to equations (29) and (30). The difficulty is due to the singularities. For $N = 0$, it suffices to use

$$\frac{1}{s_{11}^- s_{21}^-} = \frac{1}{s_{11}^- - s_{21}^-}\left(\frac{1}{s_{21}^-} - \frac{1}{s_{11}^-}\right). \tag{C.3}$$

A straightforward calculation and comparison with equation (33) gives

$$c_0^{(1)} = 0. \tag{C.4}$$

For $N = 1$, the singular terms involve fractional exponents and the steps needed are more complicated. One can employ an integral representation of the Γ function, valid for arbitrary k. It yields

$$\frac{1}{(s_{p1}^-)^k} = \frac{\mathrm{i}^k}{\Gamma(k)}\int_0^{\infty} \mathrm{d}t\, t^{k-1}\exp(-\mathrm{i}t s_{p1}^-) \tag{C.5}$$

which moves the singularities into the exponent and decouples them from the power k. Moreover, we introduce new integration variables

$$t_1 = \frac{T+\tau}{2} \quad \text{and} \quad t_2 = \frac{T-\tau}{2} \quad \text{with} \quad \tau = T\cos\vartheta. \tag{C.6}$$

All this leads to

$$\begin{aligned}
\frac{1}{(s_{11}^- s_{21}^-)^k} &= \left(\frac{\mathrm{i}^k}{\Gamma(k)}\right)^2 \int_0^{\infty} \mathrm{d}t_1 t_1^{k-1}\exp(-\mathrm{i}t_1 s_{11}^-)\int_0^{\infty} \mathrm{d}t_2 t_2^{k-1}\exp(-\mathrm{i}t_2 s_{21}^-) \\
&= \left(\frac{\mathrm{i}^k}{\Gamma(k)}\right)^2 \frac{1}{2^{2k-1}}\int_0^{\infty} \mathrm{d}T\, T^{2k-1}\exp\left(-\frac{\mathrm{i}T}{2}(s_{11}^- + s_{21}^-)\right) \\
&\quad\times \int_0^{\pi} \mathrm{d}\vartheta\, \sin^{2k-1}\vartheta\, \exp\left(-\frac{\mathrm{i}T(s_{11}-s_{21})}{2}\cos\vartheta\right) \\
&= \frac{(-1)^k\sqrt{\pi}}{2^{(2k-1)/2}\Gamma(k)}\int_0^{\infty} \mathrm{d}T\, T^{2k-1}\exp\left(-\frac{\mathrm{i}T}{2}(s_{11}^- + s_{21}^-)\right) \\
&\quad\times \frac{J_{(2k-1)/2}(T(s_{11}-s_{21})/2)}{(T(s_{11}-s_{21})/2)^{(2k-1)/2}}.
\end{aligned} \tag{C.7}$$

Here, we did the angular integral using the representation

$$\int_0^{\pi} \exp(\mathrm{i}z\cos\vartheta)\sin^{d-2}\vartheta\, d\vartheta = 2^{(d-2)/2}\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)\frac{J_{(d-2)/2}(z)}{z^{(d-2)/2}} \tag{C.8}$$

for the Bessel function in d dimensions. We now insert equation (C.7) into the implicit definition of $\omega_1^{(1)}(x_p)$ according to equations (29) and (30). We also rotate the eigenvalues

$$u = \frac{s_{11}+s_{21}}{2} \qquad \text{and} \qquad v = \frac{s_{11}-s_{21}}{2} \tag{C.9}$$

and find, at $x_p = 0$,

$$\omega_1^{(1)}(0) = \mathrm{Im} \int_{-\infty}^{+\infty} \mathrm{d}s_{11} \int_{-\infty}^{+\infty} \mathrm{d}s_{21} \frac{|s_{11} - s_{21}|}{(s_{11}^- s_{21}^-)^{3/2}} \exp\left(-\frac{1}{2}\left(s_{11}^2 + s_{21}^2\right)\right)$$
$$= \mathrm{Im} \frac{\mathrm{i}^3 2\sqrt{\pi}}{\Gamma(3/2)} \int_{-\infty}^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}v \int_0^{\infty} \mathrm{d}T\, T\, \varepsilon(v) \exp(-u^2 - v^2)$$
$$\times \exp(-\mathrm{i}Tu^-) J_1(Tv) = -8\pi + 4\sqrt{2}\pi \tag{C.10}$$

with the step function $\varepsilon(v)$ defined in equation (7). From equation (33), we also have

$$\omega_1^{(1)}(0) = -8\pi \int_0^{\infty} \exp\left(-\frac{1}{2}t^2\right) t\, \mathrm{d}t - c_1^{(1)} = -8\pi - c_1^{(1)}. \tag{C.11}$$

Hence, combining the last two formulae, we obtain

$$c_1^{(1)} = -4\sqrt{2}\pi \tag{C.12}$$

for the case $N = 1$.

Thus, we can now use the recursion (C.1) and finally arrive at the result (34).

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